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Abstract

The Aharonov–Vaidman gauge—which additively transforms the mean value of a quantum mechanical observable into an associated weak value—is introduced. It is shown that the unusual eccentric properties of weak values are inherited from this gauge and that a weak value of an observable can be considered as its mean value measured in the Aharonov–Vaidman gauge. The total time derivative of this sum also transforms an observable's mean value equation of motion into the associated weak value equation of motion and it is shown that the weak energy of evolution which influences the evolution of a weak value is intrinsic to the rate of change of the Aharonov–Vaidman gauge. Both of these equations of motion can be expressed in terms of time varying generalized coordinates and their rates of change. These equations satisfy the Euler–Lagrange equations which—in turn—define conjugate momenta and provide for their coordinate/momentum/time Poincaré representations. The underlying mathematical forms of these two representations are identical except for three symbols which distinguish them physically and identify three simple replacement operations that are required to transform the mean value Poincaré representation into the weak value Poincaré representation. This transformational relationship between Poincaré representations defines the notion of quasi-form invariance and the replacement operations encode the peculiar physical properties induced by the Aharonov–Vaidman gauge, i.e. the complexification and increased dimension of phase space, and the absorption of the weak energy of evolution by the conjugate momenta. Simple examples are used to illustrate the theory.

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1. Introduction

The weak value A_w of a quantum mechanical observable A was introduced by Aharonov *et al* [1–3] more than two decades ago. This quantity is the statistical result of a standard measurement procedure performed upon a pre-selected and post-selected (PPS) ensemble of quantum systems when the interaction between the measuring apparatus and each system is sufficiently weak. Unlike a standard strong measurement of A which significantly disturbs the measured system (i.e. it 'collapses' the wavefunction), a weak measurement of A for a PPS system does not appreciably disturb the quantum system and yields A_w as the observable's measured value. The peculiar nature of the virtually undisturbed quantum reality that exists between the boundaries defined by the PPS states is revealed by two eccentric characteristics of A_w, namely that A_w can be a complex valued quantity and that Re A_w can lie far outside the eigenvalue spectrum of the associated operator $\widehat{A}$ (hereinafter referred to as the *super-eigenlimit* property of A_w).

An additional eccentricity arises from the fact that weak value theory is a special consequence of the time-symmetric reformulation of quantum mechanics (TSQM) [3, 4]. Whereas standard quantum mechanics describes a quantum system at a time t using a state evolving forward in time from the past to t, TSQM also uses a second state evolving backward in time from the future to t. In the case of weak values, the measured value of A_w at time t not only depends upon the pre-selected forward evolving state but is also influenced by the post-selected state's backward in time evolution from the future. Although experiments performed in recent years have verified several of the unusual properties predicted by weak value theory [5–11], the interpretation of weak values remains somewhat controversial. Consequently, this apparent influence of the future post-selected state upon A_w at time t is referred to in this paper as the *quasi-nonlocal in time* property of A_w. Here the prefix 'quasi-' is used in the sense that it is 'as if' A_w depends upon this future state.

A main objective of this paper is to introduce a PPS-defined uncertainty quantity called the *Aharonov–Vaidman* (AV) *gauge* as a new 'scale of measurement' for mean values. This quantity additively transforms a mean value of an observable into an associated weak value and induces each of the eccentric characteristics exhibited by weak values, i.e. their quasi-nonlocal in time, complex valued and super-eigenlimit properties. The total time derivative of this sum also yields the weak value's equation of motion. Thus, a weak value and its equation of motion can be viewed as the associated mean value and its equation of motion expressed in the AV gauge. Here the total time derivative of the AV gauge affects the associated additive transformation of the mean value equation of motion (although it is somewhat non-standard usage, a mean value equation of motion is herein called an Ehrenfest equation).

An additional objective is to examine the weak value equation of motion from the perspective of AV gauge transformations. It is first shown that the quasi-nonlocal in time weak energy of evolution which influences the evolution of weak values [12] results from the rate of change of the AV gauge. To further develop this perspective, it is then demonstrated that—instead of its usual commutator version—an Ehrenfest equation can be expressed in terms of time varying generalized coordinates and their rates of change. Such an equation satisfies the associated Euler–Lagrange equations and defines conjugate momenta which provide for a coordinate/momentum/time Poincaré representation of the Ehrenfest equation. It is similarly shown that a weak value equation of motion also has a Poincaré representation.

These Poincaré representations share the same underlying mathematical form and differ only by three symbols which distinguish them physically in terms of phase space dimension and complexification, and weak energy of evolution absorption by conjugate momenta. This difference defines three straightforward symbol replacement operations which transform one representation to the other and introduces the notion of *quasi-form invariance*, i.e. the form invariance of the Poincaré representation of the Ehrenfest equation under AV gauge transformations. In keeping with the custom of specifying gauge transformations as 'global' or 'local', the prefix 'quasi-' is used here to refer to the quasi-nonlocal in time property of the AV gauge. This prefix is also used to denote that the complexification, increased dimension and weak energy absorption attributes of the replacement operations associated with quasi-form invariance are not inherent in the canonical notion of form invariance.

The remainder of this paper is organized as follows: in the next section, weak measurements and weak value theory are reviewed. The AV gauge is defined and discussed in section 3 and the Poincaré representation of the Ehrenfest equation is developed in section 4. Section 5 introduces AV gauging of the Ehrenfest equation, investigates the relationship between the rate of change of the AV gauge and the weak energy of evolution, discusses the absorption of the weak energy of evolution by the conjugate momenta and establishes the Poincaré representation for the weak value equation of motion. Quasi-form invariance and the symbol replacement operations are discussed in section 6. Simple examples illustrate aspects of the theory and closing remarks comprise the final section of this paper.

2. Weak measurements and weak values

Weak measurements arise in the von Neumann description of a quantum measurement at time t_0 of a time-independent observable A that describes a quantum system in an initial fixed pre-selected state $|\psi_i\rangle = \sum_J c_j|a_j\rangle$ at t_0, where the set J indexes the eigenstates $|a_j\rangle$ of $\widehat{A}$. In this description, the Hamiltonian for the interaction between the measurement apparatus and the quantum system is

$$\widehat{H} = g(t)\widehat{A}\widehat{p}.$$

Here $g(t) = g\delta(t - t_0)$ defines the strength of the measurement's impulsive interaction at t_0 and $\widehat{p}$ is the momentum operator for the pointer of the measurement apparatus which is in the initial state $|\phi\rangle$. Let $\widehat{q}$ be the pointer's position operator that is conjugate to $\widehat{p}$ and assume that $\langle q|\phi\rangle \equiv \phi(q)$ is real valued with $\langle q\rangle \equiv \langle\phi|\widehat{q}|\phi\rangle = 0$.

Prior to the measurement the pre-selected system and the pointer are in the tensor product state $|\psi_i\rangle|\phi\rangle$. Immediately following the measurement, the combined system is in the state

$$|\Phi\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int \widehat{H}\mathrm{d}t}|\psi_i\rangle|\phi\rangle = \sum_J c_j\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}ga_j\widehat{p}}|a_j\rangle|\phi\rangle,$$

where use has been made of the fact that $\int \widehat{H}\,\mathrm{d}t = g\widehat{A}\widehat{p}$. The exponential factor in this equation is the translation operator $\widehat{S}(ga_j)$ for $|\phi\rangle$ in its q-representation. It is defined by the action $\langle q|\widehat{S}(ga_j)|\phi\rangle = \langle q - ga_j|\phi\rangle \equiv \phi(q - ga_j)$ which translates the pointer's wavefunction over a distance ga_j parallel to the q-axis. The q-representation of the combined system and pointer state is

$$\langle q|\Phi\rangle = \sum_J c_j\langle q|\widehat{S}(ga_j)|\phi\rangle|a_j\rangle.$$

When the measurement interaction is strong, the quantum system is appreciably disturbed and its state 'collapses' to an eigenstate $|a_n\rangle$ leaving the pointer in the state $\langle q|\widehat{S}(ga_n)|\phi\rangle$ with probability $|c_n|^2$. Strong measurements of an ensemble of identically prepared systems yield $g\langle A\rangle \equiv g\langle\psi_i|\widehat{A}|\psi_i\rangle$ as the centroid of the pointer probability distribution

$$|\langle q|\Phi\rangle|^2 = \sum_J |c_j|^2|\langle q|\widehat{S}(ga_j)|\phi\rangle|^2, \tag{1}$$

with $\langle A\rangle$ as the measured value of $\widehat{A}$.

A weak measurement of $\widehat{A}$ occurs when the interaction strength g is small so that the system is essentially undisturbed and when the uncertainty Δq is much larger than $\widehat{A}$'s eigenvalue separation. In this case, equation (1) is the superposition of broad, strongly overlapping

$|\langle q|\widehat{S}(ga_j)|\phi\rangle|^2$ terms. Although a single measurement provides little information about $\widehat{A}$, many repetitions allow the centroid of equation (1) to be determined to any desired accuracy.

If a system state is post-selected after a weak measurement is performed, then the resulting pointer state is

$$|\Psi\rangle \equiv \langle\psi_f|\Phi\rangle = \sum_J c_j'^* c_j \widehat{S}(ga_j)|\phi\rangle,$$

where $|\psi_f\rangle = \sum_J c_j'|a_j\rangle$, $\langle\psi_f|\psi_i\rangle \neq 0$, is the post-selected state at t_0. Since

$$\widehat{S}(ga_j) = \sum_{m=0}^{\infty} \frac{[-\mathrm{i}ga_j\widehat{p}/\hbar]^m}{m!},$$

then

$$|\Psi\rangle = \sum_J c_j'^* c_j \left\{1 - \frac{\mathrm{i}}{\hbar} gA_w\widehat{p} + \sum_{m=2}^{\infty} \frac{[-\mathrm{i}g\widehat{p}/\hbar]^m}{m!}(A^m)_w\right\}|\phi\rangle \approx \left\{\sum_J c_j'^* c_j\right\}\widehat{S}(gA_w)|\phi\rangle$$

so that

$$|\langle q|\Psi\rangle|^2 \approx \left|\sum_J c_j'^* c_j\right|^2 |\langle q|\widehat{S}(g\,\mathrm{Re}\,A_w)|\phi\rangle|^2. \tag{2}$$

Here

$$(A^m)_w = \frac{\sum_J c_j'^* c_j a_j^m}{\sum_J c_j'^* c_j} = \frac{\langle\psi_f|\widehat{A}^m|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle},$$

with the weak value A_w of $\widehat{A}$ defined by

$$A_w \equiv (A^1)_w = \frac{\langle\psi_f|\widehat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}. \tag{3}$$

From this expression it is easy to see that A_w is—in general—a complex valued quantity that can be calculated directly from theory and that when $|\psi_i\rangle$ and $|\psi_f\rangle$ are nearly orthogonal $\mathrm{Re}\,A_w$ can lie far outside $\widehat{A}$'s eigenvalue spectrum, i.e. A_w has the super-eigenlimit property.

Equation (2) corresponds to a broad pointer position distribution with a single peak at $\langle q\rangle = g\,\mathrm{Re}\,A_w$ with $\mathrm{Re}\,A_w$ as the measured value of $\widehat{A}$. This occurs when both of the following relationships between g and the pointer momentum uncertainty Δp are simultaneously satisfied [13]:

$$\Delta p \ll \frac{\hbar}{g}|A_w|^{-1} \qquad \text{and} \qquad \Delta p \ll \min_{(m=2,3,\ldots)} \frac{\hbar}{g}\left|\frac{A_w}{(A^m)_w}\right|^{\frac{1}{m-1}}.$$

Note that in expressing equation (2), use has been made of the fact that since $\langle q|\phi\rangle$ is real valued, the pointer position must be translated by $g\,\mathrm{Re}\,A_w$ only. The imaginary part $\mathrm{Im}\,A_w$ influences the mean of the pointer's momentum and translates it from the initial mean by an amount proportional to the product of $\mathrm{Im}\,A_w$ with the variance of the initial pointer momentum distribution [14].

When the PPS states continuously change with time in accordance with the Schrödinger equations

$$\frac{\mathrm{d}|\psi_i\rangle}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\widehat{H}_i|\psi_i\rangle \qquad \text{and} \qquad \frac{\mathrm{d}|\psi_f\rangle}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\widehat{H}_f|\psi_f\rangle,$$

then the equation of motion for A_w is found to be

$$\dot{A}_w \equiv \frac{\mathrm{d}A_w}{\mathrm{d}t} = \frac{\mathrm{i}}{h}\{(H_fA - AH_i)_w - A_w(H_f - H_i)_w\} + \left(\frac{\partial\widehat{A}}{\partial t}\right)_w$$

(the details of the somewhat technical derivation of this equation are given in [12]). The peculiar factor $(H_f - H_i)_w$ appearing in the second term of this equation is the quasi-nonlocal in time *weak energy of evolution*. As will be shown below, this quantity is intrinsic to the rate of change of the AV gauge.

3. The Aharonov–Vaidman gauge

The action of an operator $\widehat{A}$ upon the pre-selected state $|\psi_i\rangle$ at time t can be uniquely written as [3]

$$\widehat{A}|\psi_i\rangle = \langle A\rangle|\psi_i\rangle + \Delta A\left|\psi_i^{\perp}\right\rangle, \tag{4}$$

where $\langle A\rangle = \langle\psi_i|\widehat{A}|\psi_i\rangle$ is the mean value of A and $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$ is the associated uncertainty. The contemporaneous orthonormal state $\left|\psi_i^{\perp}\right\rangle$ is $|\psi_i\rangle$'s orthogonal companion state which is induced by the action of $\widehat{A}$ upon $|\psi_i\rangle$. The companion state belongs to the subspace of $\widehat{A}$'s Hilbert space $\mathcal{H}$ that is the orthogonal complement of the subspace of $\mathcal{H}$ that contains $|\psi_i\rangle$ and it satisfies the conditions

$$\left\langle\psi_i^{\perp}\middle|\psi_i\right\rangle = 0 \qquad \text{and} \qquad \Delta A = \left\langle\psi_i^{\perp}\middle|\widehat{A}\middle|\psi_i\right\rangle. \tag{5}$$

The AV gauge is a consequence of the uncertainty/orthogonal companion state term on the right-hand side of equation (4). In particular, the AV gauge is obtained when both sides of equation (4) are multiplied from the left by the post-selected bra state $\langle\psi_f|$ at time t and then divided by the product $\langle\psi_f|\psi_i\rangle \neq 0$. The following identity for the weak value of A at time t is then obtained:

$$A_w \equiv \frac{\langle\psi_f|\widehat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \langle A\rangle + \Omega. \tag{6}$$

Thus, A_w is related to $\langle A\rangle$ by a straightforward additive transformation—the *AV transformation in gauge*$\langle\psi_f|$ (or—for short—the *AV gauge transformation*)—of $\langle A\rangle$. The additive term

$$\Omega \equiv \Delta A\frac{\left\langle\psi_f\middle|\psi_i^{\perp}\right\rangle}{\langle\psi_f|\psi_i\rangle} \tag{7}$$

is the associated *AV gauge*.

Equations (6) and (7) show that a weak value for an observable generalizes the notion of the observable's mean value in the sense that A_w can be viewed as $\langle A\rangle$ expressed or measured in AV gauge $\langle\psi_f|$ (valuewise, $A_w = \langle A\rangle$ when $\langle\psi_f| = \langle\psi_i|$ since $\Omega = 0$). Thus—from this perspective—an apparatus which measures weak values actually measures mean values in the AV gauge. It is also clear from these equations that Re A_w can lie far outside $\widehat{A}$'s eigen-spectral limits because it can be the case that $|\text{Re}\,\Omega| \to \infty$ as $\langle\psi_f| \to \left\langle\psi_i^{\perp}\right|$. Interestingly, the ratio

$$\frac{\Omega}{\Delta A} = \frac{\left\langle\psi_f\middle|\psi_i^{\perp}\right\rangle}{\langle\psi_f|\psi_i\rangle}$$

serves as a 'dissimilarity gauge' in that the more similar $|\psi_f\rangle$ is to $\left|\psi_i^{\perp}\right\rangle$, the more eccentric A_w can become. As can also be seen from this ratio, A_w is complex only when Ω is complex (obviously, Ω can be complex-valued because it is a quotient of probability amplitudes). Thus, the super-eigenlimit and complex valued eccentric characteristics of A_w are those possessed by and inherited from Ω.

In order to better understand the quasi-nonlocal in time nature of A_w and to see that this nonlocal property is also completely inherited from Ω, consider equation (7). Recall that—although the measurement of $\widehat{A}$ occurs at time t so that $|\psi_i\rangle$, $|\psi_f\rangle$ and $\left|\psi_i^{\perp}\right\rangle$ are states at t—the states $|\psi_i\rangle$ and $|\psi_f\rangle$ are actually pre-selected and post-selected at times $t_i < t$ and $t_f > t$, respectively, and $|\psi_i^{\perp}\rangle$ is induced at time t_i by the pre-selection of $|\psi_i\rangle$. If $\widehat{U}$, $\widehat{V}^{\dagger}$ and $\widehat{W}$ are unitary operators such that $|\psi_i(t)\rangle = \widehat{U}|\psi_i(t_i)\rangle$, $\langle\psi_f(t)| = \langle\psi_f(t_f)|\widehat{V}^{\dagger}$ and $|\psi_i^{\perp}(t)\rangle = \widehat{W}|\psi_i^{\perp}(t_i)\rangle$, then equation (7) can also be written equivalently as

$$\Omega = \Delta A \frac{\langle\psi_f(t_f)|\widehat{V}^{\dagger}\widehat{W}|\psi_i^{\perp}(t_i)\rangle}{\langle\psi_f(t_f)|\widehat{V}^{\dagger}\widehat{U}|\psi_i(t_i)\rangle}.$$

Observe that the action of $\widehat{V}^{\dagger}$ upon $\langle\psi_f(t_f)|$ accounts for the influence of the future post-selected state upon A_w at t. Furthermore, since both $\langle A\rangle$ and ΔA depend only upon the pre-selected state at t, then it is clear that the quasi-nonlocal in time property is strictly a characteristic of Ω and that A_w inherits this property from Ω.

Thus, A_w is an additive blending of the standard quantum mechanical properties of $\langle A\rangle$ with the non-standard eccentric properties possessed by the AV gauge. Although equation (6) states that there is a gauge freedom associated with how to choose to measure $\widehat{A}$, the measured value A_w is obviously a gauge-dependent quantity because it is contingent upon the choice made for $\langle\psi_f(t)|$ (e.g. the choice $\langle\psi_f(t)| = \langle\psi_i(t)|$ yields $A_w = \langle A\rangle$ as the measured value of $\widehat{A}$). Nevertheless, the properties of $\langle A\rangle$ are completely subsumed by those of A_w—regardless of the choice of gauge. As will be formally shown in section 5, this is also true for the dynamics of $\langle A\rangle$.

4. The Poincaré representation of the Ehrenfest equation

4.1. Theory

When expressed in the usual form

$$\langle\dot{A}\rangle = \frac{1}{\mathrm{i}\hbar}\langle[\widehat{A}, \widehat{H}]\rangle + \left\langle\frac{\partial\widehat{A}}{\partial t}\right\rangle, \tag{8}$$

the Ehrenfest equation for a quantum mechanical observable A is related to both the explicit temporality of operator $\widehat{A}$ and the commutability of $\widehat{A}$ with the associated system's Hamiltonian operator $\widehat{H}$. This form of the equation for $\langle\dot{A}\rangle$ is useful for determining if A is a constant of the motion, e.g. [15], as well as for affirming (via Ehrenfest's theorem) the formal identity between the Ehrenfest equations for quantum mechanical coordinate and conjugate momentum observables and the Hamilton equations of classical mechanics, e.g. [16].

However, when a quantum system's state depends explicitly not only upon time but also upon variables which implicitly change with time, then—as is shown here—$\langle\dot{A}\rangle$ assumes a representation that is induced by application of the chain rule for differentiation and which satisfies the associated Euler–Lagrange equations. To see this, let A be an arbitrary observable for a quantum mechanical system that is described by the normalized state

$$|\psi\rangle = \sum_J c_j(\gamma_1, \gamma_2, \ldots, \gamma_n, t)|\varphi_j\rangle \equiv \sum_J c_j|\varphi_j\rangle.$$

The set J indexes the fixed basis states $|\varphi_j\rangle$ of the system and each generalized coordinate γ_ℓ, $\ell \in L = \{1, 2, \ldots, n\}$ is real valued and assumed to change with time (here, a quantity which does not change with time is a fixed parameter—not a coordinate). Then the mean value of $\widehat{A}$ can be expressed as

$$\langle A\rangle \equiv \langle\psi|\widehat{A}|\psi\rangle = \sum_{J,K} c_j^* c_k\langle\varphi_j|\widehat{A}|\varphi_k\rangle$$

so that

$$\langle \dot{A} \rangle = \sum_{J,K} \left\{ \left[\sum_{L} \left(\frac{\partial c_j^* c_k}{\partial \gamma_\ell} \right) \dot{\gamma}_\ell + \left(\frac{\partial c_j^* c_k}{\partial t} \right) \right] \langle \varphi_j | \widehat{A} | \varphi_k \rangle + c_j^* c_k \langle \varphi_j | \frac{\partial \widehat{A}}{\partial t} | \varphi_k \rangle \right\}. \tag{9}$$

Here, $K = J$, $\widehat{A}$ depends explicitly upon time only and it is assumed that $\widehat{A}$ does not necessarily commute with $\widehat{H}$.

It is readily determined from the last equation that

$$\frac{\partial \langle \dot{A} \rangle}{\partial \gamma_r} = \sum_{J,K} \left\{ \left[\sum_{L} \frac{\partial}{\partial \gamma_r} \left(\frac{\partial c_j^* c_k}{\partial \gamma_\ell} \right) \dot{\gamma}_\ell + \frac{\partial}{\partial \gamma_r} \left(\frac{\partial c_j^* c_k}{\partial t} \right) \right] \langle \varphi_j | \widehat{A} | \varphi_k \rangle + \left(\frac{\partial c_j^* c_k}{\partial \gamma_r} \right) \langle \varphi_j | \frac{\partial \widehat{A}}{\partial t} | \varphi_k \rangle \right\}$$

and

$$\frac{\partial \langle \dot{A} \rangle}{\partial \dot{\gamma}_r} = \sum_{J,K} \left(\frac{\partial c_j^* c_k}{\partial \gamma_r} \right) \langle \varphi_j | \widehat{A} | \varphi_k \rangle, \tag{10}$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \langle \dot{A} \rangle}{\partial \dot{\gamma}_r} \right) = \sum_{J,K} \left\{ \left[\sum_{L} \frac{\partial}{\partial \gamma_\ell} \left(\frac{\partial c_j^* c_k}{\partial \gamma_r} \right) \dot{\gamma}_\ell + \frac{\partial}{\partial t} \left(\frac{\partial c_j^* c_k}{\partial \gamma_r} \right) \right] \langle \varphi_j | \widehat{A} | \varphi_k \rangle + \left(\frac{\partial c_j^* c_k}{\partial \gamma_r} \right) \langle \varphi_j | \frac{\partial \widehat{A}}{\partial t} | \varphi_k \rangle \right\}.$$

When $c_j^* c_k$, its first partial derivatives with respect to γ_ℓ and t, and its second partial derivatives with respect to γ_ℓ and γ_r and with respect to γ_r and t are continuous—as is assumed here—then

$$\frac{\partial}{\partial \gamma_\ell} \left(\frac{\partial c_j^* c_k}{\partial \gamma_r} \right) = \frac{\partial}{\partial \gamma_r} \left(\frac{\partial c_j^* c_k}{\partial \gamma_\ell} \right) \qquad \text{and} \qquad \frac{\partial}{\partial t} \left(\frac{\partial c_j^* c_k}{\partial \gamma_r} \right) = \frac{\partial}{\partial \gamma_r} \left(\frac{\partial c_j^* c_k}{\partial t} \right)$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \langle \dot{A} \rangle}{\partial \dot{\gamma}_r} \right) = \frac{\partial \langle \dot{A} \rangle}{\partial \gamma_r}. \tag{11}$$

Thus, $\langle \dot{A} \rangle$ satisfies the Euler–Lagrange equation for each generalized coordinate γ_r. Consequently—as a side note—if $C \subseteq \mathbb{R}^n$ ($\mathbb{R}$ is the set of real numbers) is the associated n-dimensional configuration space, then it must be the case that the actual path in C followed by the evolving system during a fixed time interval $[t_1, t_2]$ is such that the action $\mathcal{E} \equiv \int_{t_1}^{t_2} \langle \dot{A} \rangle \, \mathrm{d}t$ is stationary, i.e. the first variation $\delta \mathcal{E}$ vanishes with respect to path variations that vanish at the end points (clearly, this is trivially the case here since $\delta \mathcal{E} = \delta \int_{t_1}^{t_2} \mathrm{d}\langle A \rangle = \delta [\langle A \rangle]_{t_1}^{t_2} = 0$).

Equation (10) defines for each coordinate a real valued conjugate momentum p_{γ_r} according to

$$p_{\gamma_r} \equiv \frac{\partial \langle \dot{A} \rangle}{\partial \dot{\gamma}_r}. \tag{12}$$

Using this result in equation (9) yields the *Poincaré representation of the Ehrenfest equation*

$$\langle \dot{A} \rangle = \sum_{L} p_{\gamma_\ell} \dot{\gamma}_\ell + \frac{\partial \langle A \rangle}{\partial t}, \tag{13}$$

where

$$\frac{\partial \langle A \rangle}{\partial t} \equiv \sum_{J,K} \left(\frac{\partial c_j^* c_k}{\partial t} \right) \langle \varphi_j | \widehat{A} | \varphi_k \rangle + \left\langle \frac{\partial \widehat{A}}{\partial t} \right\rangle \tag{14}$$

(liberty is taken to refer to equation (13) as the Poincaré representation because $\sum_L p_{\gamma_\ell} \dot{\gamma}_\ell \, \mathrm{d}t = \sum_L p_{\gamma_\ell} \, \mathrm{d}\gamma_\ell$ is a Poincaré 1-form (e.g. [17])).

Comparison of equations (13) and (14) with equation (8) yields the identity

$$\langle[\widehat{A},\widehat{H}]\rangle = \mathrm{i}\hbar\left\{\sum_{L} p_{\gamma_\ell}\dot{\gamma}_\ell + \sum_{J,K}\left(\frac{\partial c_j^* c_k}{\partial t}\right)\langle\varphi_j|\,\widehat{A}\,|\varphi_k\rangle\right\}. \tag{15}$$

Observe that when A is a constant of the motion, i.e. $\langle\dot{A}\rangle = 0$, then $\dot{\gamma}_\ell = 0, \ell \in L$ and $\frac{\partial\langle A\rangle}{\partial t} = 0 = \left\langle\frac{\partial\widehat{A}}{\partial t}\right\rangle$ which implies that $\frac{\partial c_j^* c_k}{\partial t} = 0,\ j \in J, k \in K$. Thus,

$$\sum_{L} p_{\gamma_\ell}\dot{\gamma}_\ell + \sum_{J,K}\left(\frac{\partial c_j^* c_k}{\partial t}\right)\langle\varphi_j|\widehat{A}|\varphi_k\rangle = 0$$

which is in complete agreement with the requirement $[\widehat{A},\widehat{H}] = 0$.

4.2. Example: Spin-$\frac{1}{2}$ particle in a uniform magnetic field

As an illustration of this theory, consider the evolution of the mean value of the Pauli spin operator $\widehat{\sigma}_x$ for a spin-$\frac{1}{2}$ particle under the influence of a uniform magnetic field $\mathbf{B}$ oriented along the z-axis of a three-dimensional Cartesian reference frame. Assume that the angle θ the spin direction makes with the positive z-axis varies with time and that at any time t the normalized state for the system is

$$|\psi\rangle = \mathrm{e}^{\mathrm{i}\alpha t}\cos\frac{\theta}{2}|+\rangle + \mathrm{e}^{-\mathrm{i}\alpha t}\sin\frac{\theta}{2}|-\rangle \equiv c_1|+\rangle + c_2|-\rangle.$$

Here, $\alpha = \frac{\mu B}{\hbar}$, where μ is the magnetic moment, is a fixed parameter and $|\pm\rangle$ are the orthogonal spin basis eigenkets for the Pauli operator $\widehat{\sigma}_z$.

Using equation (9) with $\widehat{A} = \widehat{\sigma}_x$, $J = K = \{1,2\}$, $L = \{1\}$, $\gamma_1 = \theta$ and $\frac{\partial\widehat{A}}{\partial t} = \frac{\partial\widehat{\sigma}_x}{\partial t} = 0$ yields

$$\begin{aligned}\langle\dot{\sigma}_x\rangle &= \left(\dot{\theta}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial t}\right)(c_1^* c_2 + c_1 c_2^*) = \left(\dot{\theta}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial t}\right)\sin\theta\cos 2\alpha t\\ &= \dot{\theta}\cos\theta\cos 2\alpha t - 2\alpha\sin\theta\sin 2\alpha t,\end{aligned} \tag{16}$$

where use has been made of the fact that $\widehat{\sigma}_x|\pm\rangle = |\mp\rangle$. It is easily verified that this expression satisfies equation (11) and that equation (12) yields $p_\theta = \cos\theta\cos 2\alpha t$ so that equation (16) can be rewritten in its Poincaré representation as

$$\langle\dot{\sigma}_x\rangle = p_\theta\dot{\theta} + \frac{\partial\langle\sigma_x\rangle}{\partial t}, \tag{17}$$

where

$$\frac{\partial\langle\sigma_x\rangle}{\partial t} = -2\alpha\sin\theta\sin 2\alpha t.$$

Observe that these results are identical to those obtained by first computing $\langle\sigma_x\rangle \equiv \langle\psi|\widehat{\sigma}_x|\psi\rangle = \sin\theta\cos 2\alpha t$ and then taking its total time derivative.

It is also interesting to verify equation (15). Application of the time-dependent Schrödinger equation yields the following matrix representation for the system's Hamiltonian operator expressed in the $|\pm\rangle$ basis

$$\widetilde{H} = \hbar\begin{pmatrix}-\alpha & -\mathrm{i}\frac{\dot{\theta}}{2}\,\mathrm{e}^{2\mathrm{i}\alpha t}\\ \mathrm{i}\frac{\dot{\theta}}{2}\mathrm{e}^{-2\mathrm{i}\alpha t} & \alpha\end{pmatrix}.$$

Using

$$\widetilde{\sigma}_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \text{and} \qquad |\psi\rangle = \begin{pmatrix}\mathrm{e}^{\mathrm{i}\alpha t}\cos\dfrac{\theta}{2}\\ \mathrm{e}^{-\mathrm{i}\alpha t}\sin\dfrac{\theta}{2}\end{pmatrix},$$

it is readily determined that in the $|\pm\rangle$ basis representation

$$[\widehat{\sigma}_x, \widehat{H}] = \hbar \begin{pmatrix} \mathrm{i}\dot{\theta}\cos 2\alpha t & 2\alpha \\ -2\alpha & -\mathrm{i}\dot{\theta}\cos 2\alpha t \end{pmatrix}$$

and

$$\langle\dot{\sigma}_x\rangle = \frac{1}{\mathrm{i}\hbar}\langle\psi|[\widehat{\sigma}_x, \widehat{H}]|\psi\rangle = \dot{\theta}\cos\theta\cos 2\alpha t - 2\alpha\sin\theta\sin 2\alpha t = p_\theta\dot{\theta} + \frac{\partial\langle\sigma_x\rangle}{\partial t}.$$

Thus, equation (8) is in complete agreement with equation (17).

5. Gauging the Ehrenfest equation

5.1. Theory

The equation of motion for a time-dependent weak value was first introduced by Parks *et al* [6] and has been discussed more recently in terms of the influence exerted upon the evolution of a weak value by a strange quasi-nonlocal in time weak energy of evolution that is manifested by the dynamics of the PPS states [12] (thus, an apparatus which measures weak values when the associated PPS states are changing with time is also a weak energy of evolution generator). This section extends these analyses by examining the weak value equation of motion from the perspective of an AV gauge transformation of an Ehrenfest equation defined by the total time derivative of equation (6), i.e.

$$\dot{A}_w = \langle\dot{A}\rangle + \dot{\Omega}. \tag{18}$$

(Liberty is taken here to call this equation an AV gauge transformation of $\langle\dot{A}\rangle$ since transformations of this form with $\dot{A}_w$ and $\langle\dot{A}\rangle$ replaced by Lagrangian energy functions and $\dot{\Omega}$ replaced by the total time derivative of a function of coordinates and time have been historically referred to as gauge transformations in the classical mechanics literature (e.g. [18–20]).)

It is apparent from this equation that the dynamics of $\langle A\rangle$ are completely subsumed by those of A_w. Also, since the weak energy of evolution is associated only with $\dot{A}_w$ and not $\langle\dot{A}\rangle$, then it must be the case that it is intrinsic to $\dot{\Omega}$. This is made more clear by considering the special case where A_w is replaced in equation (18) by the weak energy of evolution $(H_f - H_i)_w$ and $\widehat{H}_f - \widehat{H}_i$ is assumed to be a constant of the motion so that $\frac{\mathrm{d}\langle H_f - H_i\rangle}{\mathrm{d}t} = 0$. For this special case $\mathrm{d}(H_f - H_i)_w = \mathrm{d}\Omega$, i.e. changes in the weak energy of evolution are precisely due to changes in Ω (it is interesting to note that if it is also true that $[\widehat{H}_f, \widehat{H}_i] = 0$, then $\dot{\Omega} = \frac{\mathrm{i}}{\hbar}\left[\{(H_f - H_i)^2\}_w - (H_f - H_i)_w^2\right] = \frac{\mathrm{i}}{\hbar}\Delta_w^2(H_f - H_i)$, the weak variance of the weak energy of evolution [21]).

Since $|\psi_i\rangle$ $(=|\psi\rangle)$ depends upon γ_ℓ, $\ell \in L$, and t, it is clear from equation (4) that $\left|\psi_i^\perp\right\rangle$ also depends upon these coordinates and t. Furthermore—in general—the post-selected state $|\psi_f\rangle$ introduces into the expression for Ω the additional (time varying real valued) coordinates $\gamma_{n+1}, \gamma_{n+2}, \ldots, \gamma_m$ so that

$$\Omega = \Omega(\gamma_1, \gamma_2, \ldots, \gamma_m, t)$$

and the complete collection of coordinates is now indexed by the set $L^\Omega = \{1, 2, \ldots, m\}$. Thus,

$$\dot{\Omega} = \sum_{L^\Omega}\left(\frac{\partial\Omega}{\partial\gamma_\ell}\right)\dot{\gamma}_\ell + \frac{\partial\Omega}{\partial t}. \tag{19}$$

When the first and second partial derivatives of Ω are continuous—as is assumed here—then it is readily verified that $\dot{\Omega}$ satisfies the associated Euler–Lagrange equations for each $\gamma_r, r \in L^\Omega$. Since $\langle \dot{A} \rangle$ also satisfies the Euler–Lagrange equations for each $\gamma_r, r \in L$, then—by equation (18)—$\dot{A}_w$ satisfies the Euler–Lagrange equations for each $\gamma_r, r \in L^\Omega$. Consequently—again as a side note—if $C^\Omega \subseteq \mathbb{R}^m$ is the associated m-dimensional configuration space, then it must be the case that the actual path in C^Ω followed by the system during a fixed time interval $[t_1, t_2]$ is such that the action $\mathcal{E}^\Omega \equiv \int_{t_1}^{t_2} \dot{A}_w \, \mathrm{d}t$ is stationary, i.e. the first variation $\delta\mathcal{E}^\Omega$ vanishes with respect to path variations that vanish at the endpoints (as before, this is trivially the case here since $\delta\mathcal{E}^\Omega = \delta \int_{t_1}^{t_2} \mathrm{d}A_w = \delta[A_w]_{t_1}^{t_2} = 0$). That the motion of A_w in C^Ω subsumes that of $\langle A \rangle$ in C is a consequence of the fact that since C is a subspace of C^Ω, the path followed in C is the image of that followed in C^Ω under the projection map $\pi : \mathbb{R}^m \to \mathbb{R}^n$.

The conjugate momentum $p_{\gamma_r}^\Omega$ is defined for each $r \in L^\Omega$ via

$$p_{\gamma_r}^\Omega \equiv \frac{\partial \dot{A}_w}{\partial \dot{\gamma}_r}$$

so that

$$p_{\gamma_r}^\Omega = \begin{cases} p_{\gamma_r} + \dfrac{\partial \Omega}{\partial \gamma_r}, & r \in L \\ \dfrac{\partial \Omega}{\partial \gamma_r}, & r \in L^\Omega - L, \end{cases} \tag{20}$$

where set $L^\Omega - L$ is the complement of set L with respect to set L^Ω. These momenta—along with the associated coordinates—generally increase phase space dimension and complexify it when Ω is complex valued. Also, since the weak energy of evolution is intrinsic only to $\dot{\Omega}$, then it follows from equations (19) and (20) that each $p_{\gamma_r}^\Omega$ depends upon and is defined (in part) by this weak energy, i.e. $p_{\gamma_r}^\Omega$ is said to *absorb* the weak energy of evolution via the term $\frac{\partial \Omega}{\partial \gamma_r}$. Because of this absorption, $p_{\gamma_r}^\Omega$ (and—consequently—$\dot{A}_w$) inherits from Ω the property of being quasi-nonlocal in time.

These results—and the fact that $\frac{\partial A_w}{\partial t} = \frac{\partial(\langle A \rangle + \Omega)}{\partial t}$—yield the *Poincaré representation for* $\dot{A}_w$:

$$\dot{A}_w = \sum_{L^\Omega} p_{\gamma_\ell}^\Omega \dot{\gamma}_\ell + \frac{\partial A_w}{\partial t}. \tag{21}$$

The influence of the weak energy of evolution upon $\dot{A}_w$ through its absorption by each $p_{\gamma_\ell}^\Omega$ is apparent from this equation (observe that if $\frac{\partial \Omega}{\partial t} = 0$, then the quasi-nonlocal in time property associated with $\dot{A}_w$ is entirely attributed to the conjugate momenta). It is also apparent from the comparison of equation (13) with equation (21) that the underlying mathematical form of the Poincaré representation is preserved by AV gauge transformations. However—as mentioned above and is discussed further in section 6—this preservation of form is not a form invariance in the usual sense.

5.2. Example: σ_{z_w} and $\dot{\sigma}_{z_w}$ as expressions for $\langle \sigma_z \rangle$ and $\langle \dot{\sigma}_z \rangle$ in the AV gauge

In this subsection, the Pauli spin operator $\widehat{\sigma}_z$ and the PPS states

$$|\psi_i\rangle = \cos\alpha|+\rangle + \sin\alpha|-\rangle$$

and

$$|\psi_f\rangle = \cos\beta|+\rangle + \sin\beta|-\rangle$$

at measurement time t serve to illustrate AV gauge transformation theory. Here $\widehat{\sigma}_z|\pm\rangle = \pm|\pm\rangle$, α is a time varying pre-selection angle and β a fixed post-selection angle, i.e. it is a fixed parameter—not a coordinate. Then $\langle\sigma_z\rangle \equiv \langle\psi_i|\widehat{\sigma}_z|\psi_i\rangle = \cos 2\alpha$ and $\Delta\sigma_z = \sin 2\alpha$. It is determined from equation (4) that

$$\left|\psi_i^{\perp}\right\rangle = \sin\alpha|+\rangle - \cos\alpha|-\rangle$$

and easily verified that this state satisfies conditions (5).

It follows from equation (7) that the associated AV gauge is

$$\Omega = \sin 2\alpha \tan(\alpha - \beta) \tag{22}$$

(note that here the 'dissimilarity gauge' is $\tan(\alpha-\beta)$) and from equation (6) that $\langle\sigma_z\rangle$ expressed in AV gauge $\langle\psi_f|$ is

$$\sigma_{z_w} = \cos 2\alpha + \sin 2\alpha \tan(\alpha - \beta) = \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} \tag{23}$$

(observe that $\sigma_{z_w} = \langle\sigma_z\rangle$ in AV gauge $\langle\psi_i|$). This result agrees identically with that calculated directly using the definition for σ_{z_w} obtained from equation (3). Note that although σ_{z_w} is real valued (as are $\langle\sigma_z\rangle$ and Ω), it is clear that it inherits an eccentric super-eigenlimit property from Ω ($|\langle\sigma_z\rangle| \leqslant 1$ while $|\Omega| \to \infty$ as $|\alpha-\beta| \to \frac{\pi}{2}$).

Since $\langle\dot{\sigma}_z\rangle = -2\dot{\alpha}\sin 2\alpha$ and $\frac{\partial\langle\sigma_z\rangle}{\partial t} = 0$, then

$$p_\alpha = \frac{\partial\langle\dot{\sigma}_z\rangle}{\partial\dot{\alpha}} = -2\sin 2\alpha$$

and the Poincaré representation for $\langle\dot{\sigma}_z\rangle$ is

$$\langle\dot{\sigma}_z\rangle = p_\alpha\dot{\alpha}. \tag{24}$$

Also, the total time derivative of the AV gauge is

$$\dot{\Omega} = \dot{\alpha}\{2\cos 2\alpha\tan(\alpha-\beta) + \sin 2\alpha\sec^2(\alpha-\beta)\}, \qquad \alpha - \beta \neq \frac{\pi}{2}, \frac{3\pi}{2}$$

so that application of equation (18) yields

$$\dot{\sigma}_{z_w} = \dot{\alpha}\{-2\sin 2\alpha + 2\cos 2\alpha\tan(\alpha-\beta) + \sin 2\alpha\sec^2(\alpha-\beta)\}.$$

This result is the same as that obtained by taking the total time derivative of equation (23). The last equation yields the momentum conjugate to α according to

$$p_\alpha^\Omega = \frac{\partial\dot{\sigma}_{z_w}}{\partial\dot{\alpha}} = -2\sin 2\alpha + 2\cos 2\alpha\tan(\alpha-\beta) + \sin 2\alpha\sec^2(\alpha-\beta) = p_\alpha + \frac{\partial\Omega}{\partial\alpha}.$$

Since $\frac{\partial\sigma_{z_w}}{\partial t} = 0$, then the Poincaré representation for $\dot{\sigma}_{z_w}$ is given by

$$\dot{\sigma}_{z_w} = p_\alpha^\Omega\dot{\alpha}. \tag{25}$$

The fact that this AV gauge transformation preserves the underlying mathematical form of the Poincaré representations for $\langle\dot{\sigma}_z\rangle$ and $\dot{\sigma}_{z_w}$ is especially obvious when comparing equations (24) and (25) because $\frac{\partial\langle\sigma_z\rangle}{\partial t} = \frac{\partial\sigma_{z_w}}{\partial t} = 0$ and $L^\Omega = L$.

It is interesting to use this simple example to examine how the weak energy of evolution influences the AV gauge and how it is absorbed by the conjugate momentum p_α^Ω. Application of the time-dependent Schrödinger equation to $|\psi_i\rangle$ and $|\psi_f\rangle$ yields $\widehat{H}_i = \hbar\dot{\alpha}\widehat{\sigma}_y$ and $\widehat{H}_f = \widehat{0}$, respectively, where $\widehat{\sigma}_y$ is the Pauli spin y operator. From equation (3) it is found that the associated weak energy of evolution for this example is the pure imaginary quantity given by

$$(H_f - H_i)_w = -(H_i)_w = \mathrm{i}\hbar\dot{\alpha}\tan(\alpha-\beta)$$

so that equation (22) can be rewritten as

$$\Omega = \frac{\mathrm{i}(H_i)_w}{\hbar\dot{\alpha}}\sin 2\alpha$$

(the simple relationship between $(H_i)_w$ and σ_{z_w} for this example follows from equation (6)). Then

$$\frac{\partial \Omega}{\partial \alpha} = \frac{\mathrm{i}}{\hbar \dot{\alpha}} \left\{ \left(\frac{\partial (H_i)_w}{\partial \alpha} \right) \sin 2\alpha + 2(H_i)_w \cos 2\alpha \right\}$$

and $(H_i)_w$ is absorbed by p_α^Ω according to

$$p_\alpha^\Omega = p_\alpha + \frac{\mathrm{i}}{\hbar \dot{\alpha}} \left\{ \left(\frac{\partial (H_i)_w}{\partial \alpha} \right) \sin 2\alpha + 2(H_i)_w \cos 2\alpha \right\}.$$

Since $\frac{\partial \sigma_{z_w}}{\partial t} = 0$, then it is also the case that $\dot{\sigma}_{z_w}$'s quasi-nonlocal in time property is manifested entirely by p_α^Ω through $(H_i)_w$.

6. Quasi-form invariance and symbol replacement operations

It has been demonstrated above that—in addition to inducing the quasi-nonlocal in time property—AV gauge transformations preserve the underlying mathematical form of the Poincaré representation of the Ehrenfest equation. However, casual inspection of equations (13) and (21) reveals that these equations are distinguished by the different symbols used for their summation index sets, conjugate momenta and time differentiable functions. These differences identify the following three straightforward symbol replacement operations which transform the Poincaré representation for $\langle \dot{A} \rangle$ given by equation (13) into that for $\dot{A}_w$ given by equation (21): (i) $L \rightarrow L^\Omega$, (ii) $p_{\gamma_\ell} \rightarrow p_{\gamma_\ell}^\Omega$ and (iii) $\langle A \rangle \rightarrow A_w$ (which implies $\langle \dot{A} \rangle \rightarrow \dot{A}_w$ and $\frac{\partial \langle A \rangle}{\partial t} \rightarrow \frac{\partial A_w}{\partial t}$). Here '$\rightarrow$' means replace the symbol to the left of the arrow everywhere in equation (13) with the symbol to the right of the arrow (the reverse operations are obvious).

That these replacements preserve the underlying mathematical form of equation (13) is obvious. It is also the case that they encode information about the dynamics and the physical differences between $\langle A \rangle$ and A_w. In particular, since $|L| \leqslant |L^\Omega|$, then replacement (i) can increase the dimension of the phase space for $\langle A \rangle$ in order to accommodate the additional coordinates introduced by the post-selected state (it follows that evolutionary trajectories for $\langle A \rangle$ can be injectively mapped into the phase space for A_w without perturbing the associated dynamics of $\langle A \rangle$, i.e. the dynamics for $\langle A \rangle$ are completely subsumed by those for A_w).

While replacements (ii) and (iii) clearly account for the complexification of $\dot{A}_w$ when Ω is complex valued, replacement (ii) is specifically responsible for the conjugate momenta's absorption of the weak energy of evolution. Also—as mentioned in section 2 within the context of weak measurements—complexification induced by replacements (ii) and (iii) account for the influence of $\mathrm{Im}\, \dot{A}_w$ upon the dynamics of the mean value of the measurement pointer's momentum. Taken together, replacements (i) and (ii) show that the only transformations associated with the canonical notion of form invariance that are in effect here are the coordinate transformation identities and momentum transformations $p_{\gamma_\ell}^\Omega = p_{\gamma_\ell} + \frac{\partial\, \mathrm{Re}\, \Omega}{\partial \gamma_\ell}$ performed over index set L. For the special case where $L = L^\Omega$ and complexification is not introduced by the AV gauge transformation, then—excluding the quasi-nonlocal in time property—quasi-form invariance acquires the attributes of canonical form invariance in the sense that the phase space dimension is unchanged and the usual coordinate transformation identities and momentum transformations apply. The example in section 5 is an illustration of this.

Replacements (i)–(iii), along with their physical consequences and the quasi-nonlocal in time property, are intrinsic to AV gauge transformations and characterize the quasi-form invariance of the Poincaré representation of an Ehrenfest equation under AV gauge transformations.

7. Closing remarks

The weak value for a quantum mechanical observable can be viewed as its mean value measured in the AV gauge and a weak value equation of motion can be considered to be an AV gauged Ehrenfest equation. All of the eccentric properties associated with weak values and weak value equations of motion—including the existence of the weak energy of evolution—are attributed to the AV gauge and its time derivative.

Ehrenfest equations and their AV gauged counterparts exhibit quasi-form invariant Poincaré representations. Quasi-form invariance is intrinsic to AV gauge transformations and is characterized by the induced quasi-nonlocal in time property and by three symbol replacement operations which not only maintain the underlying mathematical form of these Poincaré representations but also encode the physics induced by an AV gauge transformation of an Ehrenfest equation, i.e. increased phase space dimension, phase space complexification and absorption of the weak energy of evolution by the conjugate momenta.

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